

**$SU(N) \times S_m$ -Invariant Eigenspaces of  $N^m \times N^m$  Mean Density Matrices**

Paul B. Slater

*ISBER, University of California, Santa Barbara, CA 93106-2150**e-mail: slater@itp.ucsb.edu, FAX: (805) 893-7995*

(February 1, 2008)

We extend to additional probability measures and scenarios, certain of the recent results of Krattenthaler and Slater (quant-ph/9612043) — whose original motivation was to obtain quantum analogs of seminal work on universal data compression of Clarke and Barron. KS obtained explicit formulas for the eigenvalues and eigenvectors of the  $2^m \times 2^m$  density matrices derived by averaging the  $m$ -fold tensor products with themselves of the  $2 \times 2$  density matrices. The weighting was done with respect to a one-parameter ( $-\infty < u < 1$ ) family of probability distributions, all the members of which are spherically-symmetric ( $SU(2)$ -invariant) over the “Bloch sphere” of two-level quantum systems. For  $u = \frac{1}{2}$ , one obtains the normalized volume element of the minimal monotone (Bures) metric. In this paper, analyses parallel to those of KS are conducted, based on an alternative “natural” measure on the density matrices recently proposed by Życzkowski, Horodecki, Sanpera, and Lewenstein (quant-ph/9804024). The approaches of KS and that based on ZHSL are found to yield  $\lfloor 1 + \frac{m}{2} \rfloor$  identical  $SU(2) \times S_m$ -invariant eigenspaces (but not coincident eigenvalues for  $m > 3$ ). Companion results, based on the  $SU(3)$  form of the ZHSL measure, are presented for the twofold and threefold tensor products of the  $3 \times 3$  density matrices. In the former case, there are six invariant eigenspaces of nine-dimensional Hilbert space, and in the latter, seventeen, of twenty-seven dimensional Hilbert space. We find a rather remarkable limiting procedure (selection rule) for recovering from these analyses, the (permutationally-symmetrized) multiplets of  $SU(3)$  constructed from two or three particles. We also analyze the scenarios (all for  $m = 2$ )  $N = 4$ ,  $N = 2 \times 3$ ,  $N = 2 \times 3 \times 2$  and  $N = 3 \times 2 \times 2$  and, in addition, generalize the ZHSL measure, so that it incorporates a family of (symmetric) Dirichlet distributions — rather than simply the uniform distribution — defined on the  $(N - 1)$ -dimensional simplex of eigenvalues.

PACS Numbers 03.65.Fd, 03.67.Hk, 02.20.Qs, 05.30.-Ch

## Contents

<b>I</b>	<b>INTRODUCTION</b>	<b>2</b>
<b>II</b>	<b>THE CASE <math>N = 2</math></b>	<b>3</b>
A	The subcase $m = 1$ . . . . .	3
B	The subcases $m = 2, 3$ . . . . .	3
C	Monotone metrics . . . . .	4
D	The subcases $m = 4, 5, 6$ . . . . .	5
E	Explicit spin states . . . . .	5
<b>III</b>	<b>THE CASE <math>N = 3</math></b>	<b>7</b>
A	The subcase $m = 1$ . . . . .	7
B	The subcase $m = 2$ . . . . .	8
C	The subcase $m = 3$ . . . . .	9
D	The subcase $m = 4$ . . . . .	10
<b>IV</b>	<b>THE CASE <math>N = 2 \times 3</math></b>	<b>10</b>
A	The subcase $m = 1$ . . . . .	10
B	The subcase $m = 2$ . . . . .	10
<b>V</b>	<b>THE CASES <math>N = 2 \times 3 \times 2</math> and <math>N = 3 \times 2 \times 2</math></b>	<b>11</b>
<b>VI</b>	<b>THE CASE <math>N = 4</math></b>	<b>11</b>
<b>VII</b>	<b>THE CASE <math>N = 5</math></b>	<b>11</b>
<b>VIII</b>	<b>PARAMETERIZED FAMILIES OF <math>N^m \times N^m</math> MEAN DENSITY MATRICES</b>	<b>12</b>
A	$N = 3, m = 2$ . . . . .	12
B	$N = 4, m = 2$ . . . . .	12
C	$N = 4, m = 3$ . . . . .	13
D	$N = 2$ . . . . .	13
E	$N > 4$ . . . . .	13
<b>IX</b>	<b>CONCLUDING REMARKS</b>	<b>13</b>

## I. INTRODUCTION

Wootters, in a paper entitled “Random Quantum States,” stated that “there does not seem to be any natural measure on the set of all mixed states” [1]. Relatedly, Jones [2] asserted that the problem in the Bayesian treatment of “the more realistic experimental case of mixed input states . . . lies in selecting a good prior on mixed density matrices”. Also, somewhat earlier, Band and Park [3] were concerned with the “lack of a rational axiom of prior distribution over the entire domain of density operators”.

Contrastingly, however, Życzkowski, Horodecki, Sanpera and Lewenstein [4] have recently proposed “a natural measure in the space of density matrices” and used it to “estimate the volume of separable states, providing numerical evidence that it decreases exponentially with the dimension of the composite system”. Also, the present author, in a series of papers [5–9] (cf. [10, secs. 4 and 5] and [11]) has pursued the strategy of attempting to normalize the volume elements of certain “monotone” Riemannian metrics [13] — in particular, the minimal (Bures [14,15]) and maximal monotone ones — and use the results for information-theoretic/Bayesian, as well as thermodynamic purposes. In [8], it was concluded — supportive of certain earlier results of Petz and Toth [12] — that among the family of monotone metrics, the maximal was most *noninformative* in character.

In this communication, we seek to compare and contrast these various measures based on monotone metrics with the measures newly presented in [4]. In doing this, we immediately encounter a most interesting difference. While in [5–9], the measures are directly defined over the  $(N^2 - 1)$ -dimensional convex sets of  $N \times N$  density matrices, in [4] they are taken to be the products of the  $(N^2)$ -dimensional Haar (invariant) measure over  $U(N)$  and the uniform

measure over the  $(N - 1)$ -dimensional simplex spanned by the  $N$  (nonnegative) eigenvalues of the  $N \times N$  density matrices. Thus, in [4], the measures are defined in the (larger)  $(N^2 + N - 1)$ -dimensional space.

## II. THE CASE $N = 2$

### A. The subcase $m = 1$

We have been unable to obtain an explicit (degenerate/singular) transformation or mapping to pass from these higher-dimensional measures to the lower-dimensional ones, even for the case  $N = 2$ . Notwithstanding this, we still proceed analytically by, first, averaging the  $2 \times 2$  density matrices with respect to these differing measures. Such matrices are parameterizable, using the conventional “Bloch sphere” representation [16], as,

$$\frac{1}{2} \begin{pmatrix} 1 + r \cos \vartheta & r \sin \vartheta \cos \varphi + i r \sin \vartheta \sin \phi \\ r \sin \vartheta \cos \vartheta - i r \sin \vartheta \cos \varphi & 1 - r \cos \vartheta \end{pmatrix} \quad (1)$$

( $0 \leq r \leq 1$ ,  $0 \leq \vartheta \leq \pi$ ,  $0 \leq \phi < 2\pi$ ). Averaging these matrices with respect to any of the (normalized) measures used by Slater, expressed in the same variables, we simply obtain, as seems evident, the familiar density matrix of the *fully* mixed state,

$$\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}. \quad (2)$$

We obtain the same result using the measure of Życzkowski *et al* [4], but now expressing (1) not in terms of  $r, \vartheta$  and  $\phi$ , but rather, say, spherical polar coordinates in *four*-dimensions [17, eq. (3.129)]. We, then, perform the averaging making use of the associated Haar measure [17, eqs. (3.130), (3.131)].

### B. The subcases $m = 2, 3$

We enter a less intuitive realm apparently, however, if we average with respect to the same normalized measures (that is, probability distributions), the  $4 \times 4$  density matrices, which are the tensor products of the  $2 \times 2$  density matrices with themselves. (We note that these averaged matrices and all the ones subsequently discussed, describe by their very construction, states which are *separable*, that is, *classically correlated* [18]. For an interesting discussion of the significance of the tensor product in *quantum* computation, see [19].) In fact, in [20] (cf. [21]), Krattenthaler and Slater derived a general formula for the entries of  $2^m \times 2^m$  density matrices of this type (as well as their eigenvalues and eigenvectors), when the averaging was performed with respect to any member of the one-parameter ( $u$ ) family of probability distributions over the Bloch sphere,

$$\frac{\Gamma(\frac{5}{2} - u) r^2 \sin \vartheta}{\pi^{\frac{3}{2}} \Gamma(1 - u) (1 - r^2)^u}. \quad -\infty < u < 1 \quad (3)$$

The *asymptotics* [ $m \rightarrow \infty$ ] of the *relative entropy* of the  $2^m \times 2^m$   $m$ -fold tensor products with respect to their corresponding averages, as a function of  $u$ , was also obtained in [20], and its relevance to “universal quantum coding” discussed (cf. [22,23]).

The case  $u = \frac{1}{2}$  of (3) gives us the (normalized) volume element of the minimal (Bures) monotone metric (cf. [10, eq. (30)]). The maximal monotone metric corresponds to  $u = \frac{3}{2}$ . Its volume element is not normalizable over the *entire* Bloch sphere. But it is normalizable, if we remove an  $\epsilon$ -neighborhood of the spherical boundary ( $r = 1$ ). Then, it is possible — proceeding in Cartesian coordinates and taking the limit ( $\epsilon \rightarrow 0$ ) of a certain ratio — to obtain *two*- and *one*-dimensional (marginal) probability distributions [9]. (The one-parameter family (3) was employed in [20] for its computational tractability, in addition to its intrinsic interest. This line of work constituted an effort to develop quantum analogs of recent seminal results of Clarke and Barron concerning universal data compression [24,25].)

Now, let us, for the scenarios  $m = 2$  and  $3$ , average the  $m$ -fold tensor products with themselves of the  $2 \times 2$  density matrices, first, with respect to the (normalized) measure utilized by Życzkowski *et al*, which is, representing the Euler-Rodrigues parameters in terms of spherical polar coordinates in four dimensions ( $0 \leq \phi < 2\pi, 0 \leq \theta \leq \pi, 0 \leq \chi \leq \pi$ ), [17, eqs. (3.129)-(3.131)], proportional to

$$\sin^2 \chi \sin \theta. \quad (4)$$

The  $4 \times 4$  and  $8 \times 8$  results obtained are precisely the same, as those found by averaging the  $2^m \times 2^m$  density matrices (parameterized as in (1)) using the probability distribution for  $u = -2$  in (3). For  $m = 2$ , this common average is

$$\begin{pmatrix} \frac{5}{18} & 0 & 0 & 0 \\ 0 & \frac{2}{9} & \frac{1}{18} & 0 \\ 0 & \frac{1}{18} & \frac{2}{9} & 0 \\ 0 & 0 & 0 & \frac{5}{18} \end{pmatrix}, \quad (5)$$

(not simply the diagonal matrix with entries  $\frac{1}{4}$ , as might have been naively anticipated on the basis of (2)) and for  $m = 3$ ,

$$\begin{pmatrix} \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{9} & \frac{1}{36} & 0 & \frac{1}{36} & 0 & 0 & 0 \\ 0 & \frac{1}{36} & \frac{1}{9} & 0 & \frac{1}{36} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{9} & 0 & \frac{1}{36} & \frac{1}{36} & 0 \\ 0 & \frac{1}{36} & \frac{1}{36} & 0 & \frac{1}{9} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{36} & 0 & \frac{1}{9} & \frac{1}{36} & 0 \\ 0 & 0 & 0 & \frac{1}{36} & 0 & \frac{1}{36} & \frac{1}{9} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \end{pmatrix}. \quad (6)$$

(We observe, somewhat relatedly, that Życzkowski *et al* “could not resist the temptation to investigate the mean value of [the degree of entanglement] over random density matrices generated” according to their measure [4, App. B].)

### C. Monotone metrics

So, we are able to conclude that for the case  $N = 2$ , the measure employed by Życzkowski *et al* does not correspond to that for either the minimal or monotone metric. In fact, based on the evidence so far presented, one might conjecture that it does not correspond to any monotone metric at all. We say this based on the proposition that for any monotone metric one can associate a “Morozova-Chentsov function”, expressible as  $c(x, y) = 1/yf(x/y)$ , where  $f(1) = 1$  and  $f(t) = tf(t^{-1})$ . For the family of probability distributions (3), we have that [13, eq. (3.17)]

$$f(t) = \frac{(1+t)^{2-2u}}{2^{2-2u}t^{\frac{1}{2}-u}}. \quad (7)$$

In the minimal ( $u = \frac{1}{2}$ ) and maximal ( $u = \frac{3}{2}$ ) cases,  $f(t) = \frac{1+t}{2}$  and  $\frac{2t}{1+t}$ , respectively. Now, both of these functions are *operator monotone* — which constitutes a necessary and sufficient condition for the monotonicity of the associated metric. (A function  $f : \mathbb{R}^+ \rightarrow \mathbb{R}$  is called operator monotone if the relation  $0 \leq K \leq H$  implies  $0 \leq f(K) \leq f(H)$  for any matrices  $K$  and  $H$  of any order. Ordinary monotonicity is obtained if  $K$  and  $H$  are simply scalar quantities.) But for the choice  $u = -2$ , which yields (5) and (6), we have that

$$f(t) = \frac{(1+t)^6}{64t^{\frac{5}{2}}}, \quad (8)$$

which is not monotone (Fig. 1) nor, *a fortiori*, operator monotone. (“All stochastically monotone Riemannian metrics are characterized by means of operator monotone functions and ... there exists a maximal and minimal among them” [13].)

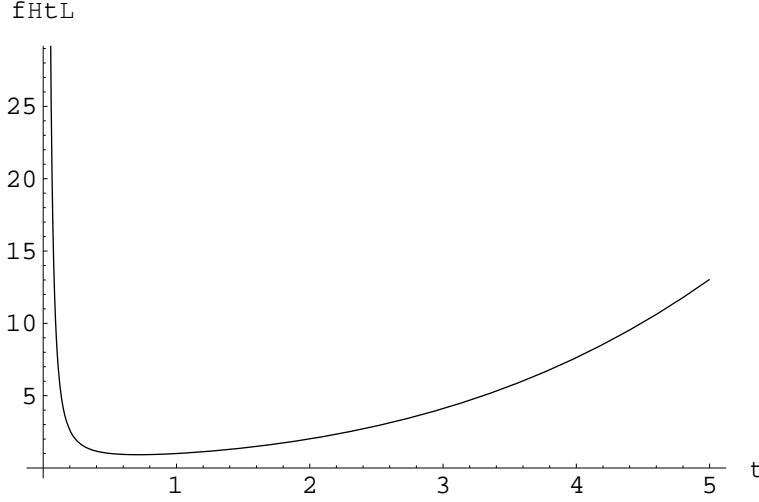


FIG. 1. Nonmonotone nature of the indicator function (8), corresponding to the probability distribution (3) with  $u = -2$

#### D. The subcases $m = 4, 5, 6$

Continuing on to the scenario  $m = 4$ , however, the legitimacy of this particular argument (sec. II C), concerning monotonicity, is undermined by the fact that the  $16 \times 16$  analog of (5) and (6) is *not* the same as that achieved using the probability distribution (3) with  $u = -2$  (or any other specific value of  $u$ ). Nevertheless, these two  $16 \times 16$  density matrices share the same zero-nonzero pattern and possess two similar sets of three distinct eigenvalues ( $\frac{7}{66}, \frac{1}{22}, \frac{1}{33}$  for the mean matrix based on (3) setting  $u = -2$ , and  $\frac{8}{75}, \frac{2}{45}, \frac{1}{30}$  for the one relying upon the measure of Życzkowski *et al*) of multiplicities five, nine and two, respectively, corresponding to *identical* eigenspaces. (C. Krattenthaler has a formal demonstration that averaging over the Bloch sphere with respect to any *spherically-symmetric* probability distribution yields the same collection of eigenspaces for any  $m$ .)

For the cases  $m = 5$  and  $6$ , we have found results of a similar nature. Using the  $U(2)$  measure of Życzkowski *et al*, the eigenvalues for  $m = 5$  are  $\frac{13}{180}, \frac{1}{40}$  and  $\frac{1}{60}$  (with multiplicities six, sixteen and ten, respectively) and for  $m = 6$ , they are  $\frac{151}{2940}, \frac{31}{2100}, \frac{11}{1260}$  and  $\frac{1}{140}$  (with multiplicities seven, twenty-five, twenty-seven and five, respectively). (These multiplicities are in conformity with the formula (10) below, reported in [20].) The corresponding averaged matrices and those based on (3) share the same zero-nonzero patterns, but do not fully match for any particular value of  $u$ . (We have not, however, explicitly confirmed that the eigenspaces are identical, as we anticipate.)

It would be quite interesting, it would appear, for its possible utility in universal quantum coding [20,23], to obtain a formula for general  $m$  for the entries and eigenvalues of the  $2^m \times 2^m$  averaged density matrix based on the measure of Życzkowski *et al*. (Presumably, the eigenvectors are, for all  $m$ , those already given by Krattenthaler and Slater [20].) Then, one could try to establish the asymptotics of the relative entropy with respect to this averaged matrix of the  $m$ -fold tensor products of the  $2 \times 2$  density matrices.

#### E. Explicit spin states

For general  $m$ , in the framework of [20] based on (3), the subspace spanned by all those eigenvectors associated with the same ( $d$ -th) eigenvalue [20, eq. (2.12)],

$$\lambda_{m,d} = \frac{1}{2^m} \frac{\Gamma(\frac{5}{2} - u)\Gamma(2 + m - d - u)\Gamma(1 + d - u)}{\Gamma(\frac{5}{2} + \frac{m}{2} - u)\Gamma(1 - u)}, \quad d = 0, 1, \dots, \lfloor \frac{m}{2} \rfloor \quad (9)$$

corresponds to those *explicit spin states* [17, sec. 7.5.j] [26–28] with  $d$  spins “up” or “down” (and the other  $m - d$  spins, of course, the reverse). (Eigenvectors — linearly independent, but not orthonormalized — are enumerated by “ballot paths” and given by formula (2.14) of [20].) The  $2^m$ -dimensional Hilbert space can be decomposed into the direct sum of carrier spaces of irreducible representations of  $SU(2) \times S_m$ . The multiplicities of the eigenvalues [20, eq. (2.13)],

$$M_{m,d} = \frac{(m-2d+1)^2}{(m+1)} \binom{m+1}{d}, \quad d = 0, 1, \dots, \lfloor \frac{m}{2} \rfloor \quad (10)$$

are the dimensions of the corresponding irreps. Thus, for  $m = 4, u = -2$ , using (9) and (10), we have the results previously mentioned,  $\lambda_{4,0} = \frac{7}{66}, \lambda_{4,1} = \frac{1}{22}, \lambda_{4,2} = \frac{1}{33}$ , and  $M_{4,0} = 5, M_{4,1} = 9, M_{4,2} = 2$ . Employing these two formulas again, we can easily ascertain that the eigenvalues of the  $4 \times 4$  matrix (5) are  $\frac{5}{18}$  (threefold) and  $\frac{1}{6}$  (unrepeated). For the  $8 \times 8$  matrix (6), we have the fourfold eigenvalue  $\frac{1}{6}$  associated with the orthonormalized eigenvectors,

$$(0, 0, 0, 0, 0, 0, 0, 1), \quad (0, 0, 0, \frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0), \quad (0, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}, 0, 0, 0), \quad (1, 0, 0, 0, 0, 0, 0, 0) \quad (11)$$

and the fourfold eigenvalue  $\frac{1}{12}$ , corresponding to

$$(0, 0, 0, -\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}, 0), \quad (0, 0, 0, -\frac{1}{\sqrt{6}}, 0, \frac{\sqrt{2}}{\sqrt{3}}, -\frac{1}{\sqrt{6}}, 0), \quad (0, -\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}, 0, 0, 0), \quad (0, -\frac{1}{\sqrt{6}}, \frac{\sqrt{2}}{\sqrt{3}}, 0, -\frac{1}{\sqrt{6}}, 0, 0, 0). \quad (12)$$

Our investigation will now move on to cases for which  $N > 2$ . However, in this regard, let us point out that in their discussion of explicit spin- $j$  states, Biedenharn and Louck [17, p. 423] have remarked that “Unfortunately, for arbitrary  $n$  and  $k$ , the construction of the basis vectors ... has never been given (fully explicitly), principally because of unsolved problems relating to the additional labels ( $\alpha$ ), which are required to specify a basis of the space  $\mathcal{V}_k^{(n)}$ , and which imply a multiplicity of occurrence of the irrep ... of  $SU(2) \times S_n$  in the representation. However, for the case  $k = \frac{1}{2}$ , it may be proved, using standard character formulas ..., that *the representation ... contains no multiply occurring irreps of  $SU(2) \times S_n$  ...* For the case of the coupling of  $n$  spin- $\frac{1}{2}$  angular momenta, the indices ( $\alpha$ ) are not required in the notation ... for the basis vectors.” (“The so-called missing label problem is a recurring one in the theory of group and Lie algebra representation theory. To be specific, suppose that  $\lambda$  labels an irrep of a group  $G$  and  $\nu$  indexes a basis for an irrep  $\kappa$  of a subgroup  $H \subset G$ . If the  $\kappa$  irreps of  $H$  that occur in the space of the  $\lambda$  irreps of  $G$  are multiplicity free, then a basis  $\{|\lambda\kappa\nu\rangle\}$  is uniquely defined by the  $(\lambda\kappa\nu)$  labels. However, if some  $\kappa$  irrep has a multiple occurrence, an additional multiplicity index  $\alpha$  is needed to label a complete basis  $\{|\lambda\alpha\kappa\nu\rangle\}$ . This is the generic situation” [29] (cf. [30]).)

In a related context, Katriel, Paldus and Pauncz [31] in an article entitled, “Generalized Dirac Identities and Explicit Relations between the Permutational Symmetry and the Spin Operators for Systems of Identical Particles,” wrote that, “the most interesting problem to be considered is associated with the breakdown of the one-to-one correspondence between the total spin and the irreducible representations of the symmetric group for systems of identical particles with an elementary spin  $\sigma > \frac{1}{2}$  ... For two particles the symmetric group is  $S_2$  with only two irreducible representations,  $[2]$  and  $[1^2]$ . For  $\sigma = \frac{1}{2}$  these two representations fully characterize the two possible total spin states  $S = 0$  and  $S = 1$ , respectively. However, for two  $\sigma = 1$  particles three different total spin states are possible, two of which are symmetric ( $[2]$ ) and one of which is antisymmetric ( $[1^2]$ ). This is the first example where we no longer have a one-to-one correspondence between the total spin states and the irreducible representations of the symmetric group. For three  $\sigma = 1$  particles the total spin is already insufficient to specify the irreducible representation; there are two different  $S = 1$  states, each belonging to a different irreducible representation. This situation is repeated, more frequently, for states of four and five particles, but it is only for six  $\sigma = 1$  particles that an even more severe labeling problem appears: namely, two different sets of functions with the same total spin ( $S = 2$ ) correspond to the same irreducible representation,  $[4, 2]$ . For  $\sigma = \frac{3}{2}$  the same labeling difficulties are encountered. Already for four such particles, more than one state with a given total spin corresponds to the same irreducible representation. For  $\sigma = 2$  this degeneracy is present for three particles already. In other words, the representation of the symmetric group generated by the spin eigenfunctions is a reducible one, and the irreducible components can have multiplicities larger than 1” (cf. [32–34]).

The abstract to this paper of Katriel, Paldus and Pauncz [31] reads: “The well-known one-to-one correspondence between the eigenstates of the total spin for a system of spin- $\frac{1}{2}$  particles and irreducible representations of the symmetric group with up to two rows in the Young shape is the basis of interesting formal developments in quantum chemistry and in the theory of magnetism. As an explicit manifestation of this correspondence the class operators of the symmetric group are demonstrated to be expressible in terms of the total spin operator. This correspondence does not hold for higher elementary spins. The extension to arbitrary spin is investigated using Schrödinger’s generalization

of the Dirac identity, which expresses the transposition in terms of two-particle spin operators. It is shown that additional operators, which for  $\sigma = \frac{1}{2}$  reduce to the total spin operator, are needed for a complete classification. Some aspects of the formalism are developed in detail for  $\sigma = 1$ . In this case a classification identical with that provided by the irreducible representations of the symmetric group is obtained in terms of the eigenstates of two commuting operators, one of which is the total spin operator.”

It appears that the research reported below is somewhat similar in nature to that of Katriel, Paldus, and Pauncz, but with the focus now on the eigenstates not of the  $N \times N$  operators  $O_k$  (in their notation), where [31, eq. (38)] ( $s_i = (s_{xi}, s_{yi}, s_{zi})$  being a one-particle spin operator)

$$O_k = O_k(N) = \sum_{i < j}^N (s_i s_j)^k, \quad (13)$$

but on the eigenstates of the  $N^m \times N^m$  mean density matrix obtained by averaging with respect to the (normalized)  $U(N)$ -invariant measure. (“We have not been able to work out the general commutation relation between  $O_k$  and  $O_l$  for arbitrary  $k$  and  $l$  larger than 1, but we would like to conjecture that  $O_n; n = 1, 2, \dots, 2\sigma$  form a set of commuting operators for arbitrary  $\sigma$ . The missing proof of this conjecture is the most obvious loose end in the present investigation” [31].)

### III. THE CASE $N = 3$

#### A. The subcase $m = 1$

We have performed a series of computations in the framework of Życzkowski *et al* [4] for the case  $N = 3$ , utilizing the *Euler-angle* parameterization of  $SU(3)$ , along with the invariant volume element, recently given by Byrd [35, eq. (1) and p. 14] (cf. [36, eq. (17)] and [37, p. 5]). Averaging the  $3 \times 3$  density matrices accordingly, we obtained the simple intuitive result (cf. (2)),

$$\begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}. \quad (14)$$

We have not, for the  $3 \times 3$  density matrices been able to perform a strictly comparable averaging, utilizing the normalized volume elements of the maximal and monotone metrics, but certain interesting results based on them have, nevertheless, been obtained [6,9]. (We first note, analogously to the  $2 \times 2$  density matrices, that the integral over the *entire* eight-dimensional convex set of the volume element of the maximal monotone metric *diverges*. For the minimal monotone metric, on the other hand, such divergence appears not to occur. However, it is more problematical in nature, in that it appears not possible to symbolically integrate the volume element completely over the eight dimensions [9].) In particular, in [9], using a *double-limiting* argument (necessitated by the non-normalizability of the volume element of the maximal monotone metric), a *six-dimensional marginal* probability distribution was found based on a set of eight (separable) variables parameterizing the  $3 \times 3$  density matrices. Its *two-dimensional* marginal probability over the simplex spanned by the diagonal entries ( $a, b, c$ ) was

$$\frac{15(1-a)\sqrt{a}}{4\pi\sqrt{b}\sqrt{c}}. \quad (15)$$

The original symmetry between these three variables was *broken* by a series of transformations — suggested by work of Bloore [38] — employed to obtain the eight variables, the separation of which was required in order to perform the necessary integrations to obtain (15). It is, then, natural to associate the variable  $a$  with the *middle* of the three levels (the one inaccessible to a spin-1 photon, due to its masslessness). The expected values with respect to (15) are  $\langle a \rangle = \frac{3}{7}, \langle b \rangle = \langle c \rangle = \frac{2}{7}$ . We, of course, note that neither of these values equals  $\frac{1}{3}$  as in (14), based on the approach of Życzkowski *et al* [4].

## B. The subcase $m = 2$

We have also conducted the same form of averaging as used to get (14) for the  $9 \times 9$  density matrices (the “two-trit” case [39–41]) obtained by taking the twofold tensor products ( $m = 2$ ) of the  $3 \times 3$  density matrices ( $N = 3$ ). The result obtained is the symmetric matrix (the  $U(3)$  analog of the  $4 \times 4$  matrix (5), that was based on  $U(2)$  (cf. [17, p. 423]))

$$\begin{pmatrix} \frac{1}{8} & 0 & g & 0 & 0 & \frac{10g}{3} & g & \frac{10g}{3} & 0 \\ 0 & \frac{5}{48} & -2g & \frac{1}{48} & 0 & -2g & -2g & -2g & 0 \\ g & -2g & \frac{5}{48} & -2g & \frac{10g}{3} & 0 & \frac{1}{48} & 0 & g \\ 0 & \frac{1}{48} & -2g & \frac{5}{48} & 0 & -2g & -2g & -2g & 0 \\ 0 & 0 & \frac{10g}{3} & 0 & \frac{1}{8} & g & \frac{10g}{3} & g & 0 \\ \frac{10g}{3} & -2g & 0 & -2g & g & \frac{5}{48} & 0 & \frac{1}{48} & g \\ g & -2g & \frac{1}{48} & -2g & \frac{10g}{3} & 0 & \frac{5}{48} & 0 & g \\ \frac{10g}{3} & -2g & 0 & -2g & g & \frac{1}{48} & 0 & \frac{5}{48} & g \\ 0 & 0 & g & 0 & 0 & g & g & g & \frac{1}{8} \end{pmatrix}, \quad (16)$$

where  $g = \frac{1}{864\pi} \approx .000368414$ . (The off-diagonal entries,  $\frac{1}{48}$ , lacking the constant  $\pi$ , correspond to the expected value of the product of symmetrically located off-diagonal entries —  $a_{ij}a_{ji}$  — in the underlying  $3 \times 3$  density matrix  $[a_{ij}]$ .) Tracing the  $9 \times 9$  density matrix (16) over one of the two constituent subsystems, we obtain the  $3 \times 3$  diagonal density matrix (14) with entries equal to  $\frac{1}{3}$ .

Of the nine eigenvalues of (16), three are  $\frac{1}{12}$ , two are  $\frac{1}{8}$ , and the other four (unrepeated) ones can be paired as

$$\lambda_6 = \frac{1}{8} + \frac{7}{1296\pi\sqrt{2}} \approx .126216, \quad \lambda_7 = \frac{1}{8} - \frac{7}{1296\pi\sqrt{2}} \approx .123784, \quad (17)$$

and

$$\lambda_8 = \frac{1}{8} + \frac{\sqrt{331}}{1296\pi\sqrt{2}} \approx .12816, \quad \lambda_9 = \frac{1}{8} - \frac{\sqrt{331}}{1296\pi\sqrt{2}} \approx .12184.$$

The three-dimensional subspace is spanned by the orthonormal set of eigenvectors

$$(0, 0, 0, 0, 0, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0), \quad (0, 0, -\frac{1}{\sqrt{2}}, 0, 0, 0, \frac{1}{\sqrt{2}}, 0, 0), \quad (0, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0, 0, 0, 0, 0), \quad (18)$$

while the two-dimensional subspace is generated by

$$(0, \frac{1}{3\sqrt{2}}, 0, \frac{1}{3\sqrt{2}}, 0, 0, 0, 0, \frac{2\sqrt{2}}{3}), \quad (\frac{9}{\sqrt{331}}, \frac{26}{3\sqrt{331}}, 0, \frac{26}{3\sqrt{331}}, \frac{9}{\sqrt{331}}, 0, 0, 0, -\frac{13}{3\sqrt{331}}). \quad (19)$$

The remaining four one-dimensional eigenspaces (corresponding to the sequence of eigenvalues (17), which can be viewed as providing “labels” to the eigenspaces) are given by

$$(-\frac{1}{2}, 0, -\frac{1}{2\sqrt{2}}, 0, \frac{1}{2}, \frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, 0), \quad (\frac{1}{2}, 0, -\frac{1}{2\sqrt{2}}, 0, -\frac{1}{2}, \frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, 0), \quad (20)$$

$$(\frac{13}{2\sqrt{331}}, -\frac{6}{\sqrt{331}}, -\frac{1}{2\sqrt{2}}, -\frac{6}{\sqrt{331}}, \frac{13}{2\sqrt{331}}, -\frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}}, \frac{3}{\sqrt{331}}),$$

$$(\frac{13}{2\sqrt{331}}, -\frac{6}{\sqrt{331}}, \frac{1}{2\sqrt{2}}, -\frac{6}{\sqrt{331}}, \frac{13}{2\sqrt{331}}, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{3}{\sqrt{331}}).$$

We note that the constant  $\pi$ , appearing throughout the  $9 \times 9$  averaged matrix (16) and in the four eigenvalues (17), is absent — similarly to the parameter  $u$  in the  $2^m \times 2^m$  analyses of Krattenthaler and Slater [20] — from the set of nine orthonormalized eigenvectors (18), (19) and (20). Thus, we are able, by replacing  $\pi$  in (16) by a free parameter ( $v$ ), to obtain a one-parameter family of  $9 \times 9$  matrices, all the members of which have the same sets of eigenvectors



(18), (19) and (20). (We might speculate that there exists a family of probability distributions, parameterized by  $v$ , which would give these  $9 \times 9$  density matrices, as their expected values, as we now know is, in fact, the case for the particular value  $v = \pi$  [cf. sec. VIII].) For  $|v| \geq \frac{\sqrt{331}}{162\sqrt{2}} \approx .0794116$ , the  $9 \times 9$  matrix has nonnegative eigenvalues, so it is, then, a *density* matrix. If one sets  $v$  to either  $\frac{\sqrt{331}}{54\sqrt{2}} \approx .238235$  or  $\frac{7}{54\sqrt{2}} \approx .091662$ , two of the unrepeated eigenvalues become  $\frac{1}{12}$  and  $\frac{1}{6}$ . As  $v$  tends to  $\pm\infty$ , the four unrepeated eigenvalues (17) all approach  $\frac{1}{8}$ . So, in that limit, we obtain a sextet and an “antitriplet” [42, p. 311]. The antitriplet is spanned by the vectors (18), while (replacing (19) and (20)), the sextet is spanned by

$$\begin{aligned} (0, 0, 0, 0, 0, 0, 0, 0, 1), & \quad (0, 0, 0, 0, 0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0), & \quad (0, 0, \frac{1}{\sqrt{2}}, 0, 0, 0, \frac{1}{\sqrt{2}}, 0, 0), \\ (0, 0, 0, 0, 1, 0, 0, 0, 0), & \quad (0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0, 0, 0, 0, 0), & \quad (1, 0, 0, 0, 0, 0, 0, 0, 0). \end{aligned} \quad (21)$$

“We know that the product of two  $SU(3)$  triplets decomposes into a sextet and an antitriplet. The sextet is symmetric, whereas the antitriplet is antisymmetric. Therefore, both representations can be contained with the baryon multiplet which has mixed symmetry, but only the sextet can be contained in the multiplet of baryon resonances which corresponds to a totally symmetric representation” [42, p. 388].

### C. The subcase $m = 3$

For the subcase  $m = 3$  of  $N = 3$ , the diagonal entries of the  $27 \times 27$  density matrix, obtained by averaging according to the measure of Życzkowski *et al* [4], are

$$(\alpha, \beta, \beta, \beta, \beta, \gamma, \beta, \gamma, \beta, \beta, \beta, \beta, \gamma, \beta, \alpha, \beta, \gamma, \beta, \beta, \beta, \gamma, \beta, \beta, \beta, \beta, \alpha), \quad (22)$$

where  $\alpha = \frac{31}{600} \approx .0516667$ ,  $\beta = \frac{11}{300} \approx .0366667$ ,  $\gamma = \frac{37}{1200} \approx .030833$ . We have been able to determine the *off*-diagonal entries (of which, 272 are zero), as well. For instance, the (1,3) entry along with twenty-three others have the value  $\frac{7}{8640\pi}$ . Of the non-zero off-diagonal entries, most were of such a form, that is  $\frac{k}{l\pi}$ . However, there were also three distinct *rational* numbers:  $\frac{3}{400}$  (thirty-six occurrences),  $\frac{7}{1200}$  (eighteen occurrences) and  $\frac{1}{600}$  (twelve occurrences). If we denote the entries of the underlying  $3 \times 3$  density matrix by  $a_{ij}$  ( $i, j = 1, 2, 3$ ), then the cells in the  $27 \times 27$  density matrix yielding  $\frac{3}{400}$  corresponded to entries of the form  $a_{ii}a_{ij}a_{ji}$ , those yielding  $\frac{7}{1200}$  corresponded to entries of the form  $a_{ii}a_{jk}a_{kj}$ , and those yielding  $\frac{1}{600}$  were associated with cells of the type  $a_{ij}a_{jk}a_{ki}$  (where  $i, j, k$  now refer to *distinct* values of 1, 2 or 3).

Seven of the twenty-seven eigenvalues were expressible as rational numbers. These were

$$\frac{1}{60} \approx .0166667 \quad (\text{isolated}), \quad \frac{7}{240} \approx .0291667 \quad (\text{fourfold}), \quad \frac{31}{600} \approx .0516667 \quad (\text{twofold}). \quad (23)$$

Among the remaining eigenvalues, there were three pairs, corresponding to the roots ( $\lambda$ ) of the sixth-degree even polynomial, where  $y = (7 - 240\lambda)\pi$ ,

$$5504 - 317043y^2 + 4986360y^4 - 19131876y^6 = 0. \quad (24)$$

The eigenvalues ( $\lambda$ ) are *explicitly* expressible as

$$\frac{7}{240} - \frac{1}{720\pi\sqrt{2}} \approx .0288541, \quad \frac{7}{240} + \frac{1}{720\pi\sqrt{2}} \approx .0294793, \quad (25)$$

$$\frac{7}{240} - \frac{1}{405\pi\sqrt{2}} \approx .0286109, \quad \frac{7}{240} + \frac{1}{405\pi\sqrt{2}} \approx .0297224,$$

and

$$\frac{7}{240} - \frac{\sqrt{43}}{6480\pi\sqrt{2}} \approx .0289389, \quad \frac{7}{240} + \frac{\sqrt{43}}{6480\pi\sqrt{2}} \approx .0293994.$$

There were also eight isolated eigenvalues ( $\lambda = .0495426, .0496514, .0500807, .0515902, .0517431, .0532526, .0536819, .0537907$ ), corresponding to the roots of the eighth-degree even polynomial, where  $y = (31 - 600\lambda)\pi$ ,

$$383127080633857021 - 18544605647405907654y^2 + 4501753947892101744y^4 \quad (26)$$

$$-351843587054438400y^6 + 8926168066560000y^8 = 0.$$

The smallest and largest of these eight isolated eigenvalues are

$$\frac{31}{600} - \frac{\sqrt{2337181}}{162000\pi\sqrt{2}} \approx .0495426, \quad \frac{31}{600} + \frac{\sqrt{2337181}}{162000\pi\sqrt{2}} \approx .0537907. \quad (27)$$

If we replace the constant  $\pi$  by the variable  $v$  (as was done in the  $N = 3, m = 2$  case of sec. III B [cf. (21)]), then the seven rational eigenvalues are unchanged and the other twenty can be obtained from the solutions ( $\lambda$ ) of (24) and (26) using now  $y = (7 - 240\lambda)v$  and  $y = (31 - 600\lambda)v$ . In the limit  $v \rightarrow \pm\infty$ , the eigenvalues of the matrix degenerate to  $\frac{1}{60}$  (isolated),  $\frac{7}{240}$  (sixteenfold) and  $\frac{31}{600}$  (tenfold), in agreement with the cardinalities of the multiplets of  $SU(3)$  constructed from three particles [42, exer. 9.5], [43, eq. (11.5.4)].

The eigenvector corresponding to the lowest-lying (isolated) eigenvalue ( $\frac{1}{60}$ ) is

$$(0, 0, 0, 0, 0, -a, 0, a, 0, 0, 0, a, 0, 0, 0, -a, 0, 0, 0, -a, 0, a, 0, 0, 0, 0, 0), \quad (28)$$

where  $a = \frac{1}{\sqrt{6}}$ .

#### D. The subcase $m = 4$

We have ascertained that on the diagonal of the  $81 \times 81$  density matrix averaged according to the measure of Życzkowski *et al* [4], there are three occurrences of  $\frac{7}{300} \approx .0233333$ , eighteen of  $\frac{11}{900} \approx .0122222$ , twenty-four of  $\frac{17}{1200} \approx .0141667$  and thirty-six of  $\frac{37}{3600} \approx .0102778$ . We have only so far, however, been able to determine a relatively small number of the off-diagonal entries. For example, the (1,3)-entry has the value  $\frac{41}{86400\pi}$ , the (2,3)-entry,  $-\frac{43}{54000\pi}$ , and the (11,12)-entry,  $-\frac{37}{108000\pi}$ .

### IV. THE CASE $N = 2 \times 3$

#### A. The subcase $m = 1$

Let us consider the tensor product of an arbitrary  $2 \times 2$  density matrix and an arbitrary  $3 \times 3$  density matrix (cf. [44]). Then, we average the  $6 \times 6$  result (corresponding to the subcase  $m = 1$ ) over the convex sets of two-dimensional and three-dimensional density matrices, using the  $U(2) \times U(3)$  product measure, in the natural extension of the work of Życzkowski *et al* [4]. We obtain a diagonal matrix with entries  $\frac{1}{6}$ , which is simply the same as the tensor product of (2) and (14), evidence of the statistical independence of the constituent density matrices.

#### B. The subcase $m = 2$

Now, however, let us similarly average the twofold tensor products ( $m = 2$ ) of these  $6 \times 6$  density matrices with themselves. Then, the eigenvalues of the averaged  $36 \times 36$  density matrix (966 of the 1296 entries being zero) are  $\frac{1}{72} \approx .0138889$  (multiplicity three),  $\frac{1}{48} \approx .020833$  (two),  $\frac{5}{216} \approx .0231481$  (nine),  $\frac{5}{144} \approx .0347222$  (six) and the remaining ones (all explicitly expressible as fractions involving square roots and  $\pi$ ) are .0203067 (one), .0206307 (one), .021036 (one), .0213599 (one), .0338445 (three), .0343845 (three), .0350599 (three) and .0355999 (three). For instance, the last of these values is an approximation (cf. (17)) to  $\frac{5}{144} + \frac{5\sqrt{331}}{23328\pi\sqrt{2}}$ . If, as previously, we replace the occurrences of  $\pi$  in (the denominators of certain entries of) the averaged matrix by a parameter  $v$  and let  $v \rightarrow \infty$  (or, equivalently, simply set the entries in question to zero), we obtain the eigenvalues  $\frac{1}{72}$  (multiplicity three),  $\frac{1}{48}$  (six),  $\frac{5}{216}$  (nine) and  $\frac{5}{144}$  (eighteen). For the first of the corresponding four eigenspaces, all the nonzero entries of the (three) orthonormalized spanning eigenvectors were  $\pm\frac{1}{2}$ , for the second and third eigenspaces,  $\pm\frac{1}{2}$  or  $\pm\frac{1}{\sqrt{2}}$ , and for the (dominant) fourth, the entries were 1 or  $\frac{1}{2}$  or  $\frac{1}{\sqrt{2}}$ .

## V. THE CASES $N = 2 \times 3 \times 2$ AND $N = 3 \times 2 \times 2$

Let us continue further along the lines of the immediately preceding analysis (sec. IV). We construct a  $12 \times 12$  density matrix by taking the ordered tensor product of arbitrary  $2 \times 2$ ,  $3 \times 3$  and  $2 \times 2$  ones. Then, by taking the tensor product of the result with itself, we obtain a  $144 \times 144$  density matrix, which we average with respect to the product measure for  $U(2) \times U(3) \times U(2)$ . The mean density matrix obtained had 16,886 of its 20,736 entries, zero. Let us, first, report the structure of the eigenvalues of this matrix, if we replace the occurrences in its off-diagonal cells of  $\pi$  by a parameter  $v$ , which we then let go to infinity, as we have previously in this series of analyses. There are, then, six eigenspaces, corresponding to the eigenvalues  $\frac{1}{360}$  (multiplicity six),  $\frac{1}{270}$  (twenty-seven),  $\frac{1}{240}$  (twelve),  $\frac{1}{180}$  (fifty-four),  $\frac{2}{225}$  (fifteen),  $\frac{1}{75}$  (thirty).

Leaving the constant  $\pi$  unaltered, we obtain eighteen, rather than six, eigenspaces. The multiplicities of the eigenvalues  $\frac{1}{360}$ ,  $\frac{1}{270}$  and  $\frac{2}{225}$  are the same as in the  $v \rightarrow \infty$  analysis, while that of  $\frac{1}{240}$  is reduced from twelve to four,  $\frac{1}{180}$  from fifty-four to eighteen and  $\frac{1}{75}$  from thirty to ten. The manner in which these reductions in dimensionality take place can be seen by examining the orthogonal 64-dimensional Hilbert space. It is composed of four eigenspaces of dimension nine (corresponding to the eigenvalues  $\frac{1}{180} \pm \frac{7}{29160\pi\sqrt{2}}$  and  $\frac{1}{180} \pm \frac{\sqrt{331}}{29160\pi\sqrt{2}}$ ), four of dimension five (associated with the eigenvalues  $\frac{1}{75} \pm \frac{\sqrt{331}}{12150\pi\sqrt{2}}$  and  $\frac{1}{75} \pm \frac{7}{12150\pi\sqrt{2}}$ ) and four of dimension two (corresponding to the eigenvalues  $\frac{1}{240} \pm \frac{7}{38880\pi\sqrt{2}}$  and  $\frac{1}{240} \pm \frac{\sqrt{331}}{38880\pi\sqrt{2}}$ ). Of course, it is easily noted that  $4 \times 9 = 54 - 18$ ,  $4 \times 5 = 30 - 10$  and  $4 \times 2 = 12 - 4$ .

We have also conducted a parallel series of analyses for  $N = 3 \times 2 \times 2$ , thus, ordering the three density matrices in the initial tensor product differently. We, of course, again obtain a  $144 \times 144$  mean density matrix (having, once again, 16,886 of its 20,736 entries, equal to zero). In the limit,  $v \rightarrow \infty$ , an eigenanalysis yielded precisely the same results as the corresponding analysis in the  $2 \times 3 \times 2$  case. And in fact, leaving the constant  $\pi$  unaltered, we obtained the same set of eighteen distinct eigenvalues and associated multiplicities as in that case too.

## VI. THE CASE $N = 4$

For this analysis, we rely upon the (Hurwitz/Euler-angle) parameterization, together with the accompanying Haar measure, for  $U(N)$ , for general  $N$ , given by Życzkowski and Kuś [45, eqs. (3.1) -(3.5)]. For the subcase  $m = 1$ , the averaged  $4 \times 4$  diagonal matrix has non-zero entries equal to  $\frac{1}{4}$ . For the subcase  $m = 2$ , the averaged matrix had the diagonal entries

$$(\alpha, \beta, \gamma, \kappa, \beta, \alpha, \gamma, \kappa, \gamma, \gamma, \epsilon, \zeta, \kappa, \kappa, \zeta, \eta), \quad (29)$$

where  $\alpha = \frac{2327}{32400}$ ,  $\beta = \frac{3947}{64800}$ ,  $\gamma = \frac{1759}{28800}$ ,  $\kappa = \frac{971}{17280}$ ,  $\epsilon = \frac{1583}{21600}$ ,  $\zeta = \frac{2357}{43200}$  and  $\eta = \frac{299}{3600}$ . The only non-zero off-diagonal entries (in the *symmetric* mean  $16 \times 16$  density matrix) were:  $\frac{707}{64800}$  in the (2,5)-cell;  $\frac{319}{28800}$  in the (3,9) and (7,10)-cells;  $\frac{107}{17280}$  in the (4,13) and (8,14)-cells; and  $\frac{197}{43200}$  in the (12,15)-cell. We note the absence of the constant  $\pi$ , in contrast to the scenarios of secs. III, IV and V, in which  $3 \times 3$  density matrices were incorporated.

There were seven distinct eigenvalues :  $\frac{1}{20} = .05$  (having multiplicity six),  $\frac{1277}{21600} \approx .0591204$  (isolated),  $\frac{539}{8640} \approx .0623843$  (two),  $\frac{2327}{32400} \approx .071821$  (three),  $\frac{1039}{14400} \approx .0721528$  (two),  $\frac{1583}{21600} \approx .073287$  (isolated) and  $\frac{299}{3600} \approx .0830556$  (isolated) (cf. [42, exer. 11.6, eq. (1)]). (The multiplets of a two-particle systems in the group  $SU(4)$  are a *sextet* and a decuplet [42, ex. 11.8(1)].) Orthonormal bases for the seven eigenspaces are easily constructed, in which each eigenvector has fourteen or fifteen components zero, and the others either equal to 1 or  $\pm \frac{1}{\sqrt{2}}$ .

## VII. THE CASE $N = 5$

For the case,  $N = 5, m = 2$ , we have only so far been able to determine that the first two diagonal entries of the mean  $25 \times 25$  density matrix are  $\frac{2}{45}$  and  $\frac{7}{180}$  (which must also be the value of the sixth diagonal entry). We did in fact pursue a parallel (and more computationally manageable) analysis, in which the *5times5 unitary* matrices were replaced by the  $5 \times 5$  *orthogonal* matrices, and the appropriate Haar measure [45, App. A], then, used. However, the mean density matrix for the case  $m = 1$  was found to be null.

### VIII. PARAMETERIZED FAMILIES OF $N^M \times N^M$ MEAN DENSITY MATRICES

In our previous analyses, we have employed the measures introduced by Życzkowski *et al* [4] to find, for a specific choice of the dimension  $N$  and power  $m$ , a particular mean density matrix. However, it should be noted that the imposition by Życzkowski *et al* of a *uniform* measure on the  $(N-1)$ -dimensional simplex spanned by the eigenvalues  $(e_1, e_2, \dots, e_N)$  is somewhat arbitrary in nature. In fact, Bayesian principles would suggest that one might employ instead a Dirichlet distribution,

$$\frac{\Gamma(N - q_1 - q_2 - \dots - q_N)}{\Gamma(1 - q_1)\Gamma(1 - q_2)\dots\Gamma(1 - q_N)} e_1^{-q_1} e_2^{-q_2} \dots e_N^{-q_N} \quad (30)$$

with its  $N$  parameters  $(q_1, q_2, \dots, q_N)$  all set to  $\frac{1}{2}$  [46, eq. (3.7)]. (The uniform distribution is obtained by setting the parameters all to zero.) This observation has led us to expand our analysis to include (symmetric) Dirichlet distributions with all  $N$  parameters set equal to  $q$ . (We also note that in the analysis of Krattenthaler and Slater [20] a one-parameter ( $u$ ) family of probability distributions (3) was utilized to obtain a one-parameter family of  $2^m \times 2^m$  mean density matrices.)

#### A. $N = 3, m = 2$

We have been able to implement this approach for the case  $N = 3, m = 2$  of sec. IIIB. All the members of the one-parameter family of  $9 \times 9$  density matrices obtained possess the same zero-nonzero pattern as (16). The diagonal entries  $\frac{1}{8}$  and  $\frac{5}{48}$  (corresponding, as noted, to  $q = 0$ ) are replaced, now, by the more general entries,  $\frac{3-2q}{6(4-3q)}$  and  $\frac{5-4q}{12(4-3q)}$ , respectively. The constant  $g$  is replaced by  $\frac{1}{216\pi(4-3q)}$  and the off-diagonal entry,  $\frac{1}{48}$  by  $\frac{1}{12(4-3q)}$ . Instead of the three eigenvalues  $\frac{1}{12}$ , we have  $\frac{1}{9} - \frac{1}{9(4-3q)}$  and in place of the double eigenvalue  $\frac{1}{8}$ , there is  $\frac{1}{9} + \frac{1}{18(4-3q)}$ . The two pairs of unrepeated eigenvalues (17) take the more general form,

$$\lambda_6 = \frac{3-2q}{6(4-3q)} + \frac{7}{324(4-3q)\pi\sqrt{2}}, \quad \lambda_7 = \frac{3-2q}{6(4-3q)} - \frac{7}{324(4-3q)\pi\sqrt{2}}, \quad (31)$$

$$\lambda_8 = \frac{3-2q}{6(4-3q)} + \frac{\sqrt{331}}{324(4-3q)\pi\sqrt{2}}, \quad \lambda_9 = \frac{3-2q}{6(4-3q)} - \frac{\sqrt{331}}{324(4-3q)\pi\sqrt{2}}.$$

The corresponding eigenspaces are precisely the same as those given by (18)-(20).

#### B. $N = 4, m = 2$

We have pursued an analogous strategy for the  $N = 4, m = 2$  analysis of sec. VI, setting the four parameters of the Dirichlet distribution (30), again, all to  $q$ . The zero-nonzero pattern was the same as previously reported (that is, for the particular instance,  $q = 0$ ). Then, the diagonal entries — as denoted in (29) — took the form,

$$\alpha = \frac{2327 - 1620q}{6480(5-4q)}, \quad \beta = \frac{3947 - 3240q}{12960(5-4q)}, \quad \gamma = \frac{1759 - 1440q}{5760(5-4q)}, \quad \kappa = \frac{971 - 864q}{3456(5-4q)}, \quad (32)$$

$$\epsilon = \frac{1583 - 1080q}{4320(5-4q)}, \quad \zeta = \frac{2357 - 2160q}{8640(5-4q)}, \quad \eta = \frac{299 - 180q}{720(5-4q)}.$$

The (2,5)-cell is now  $\frac{707}{12960(5-4q)}$ , while the (3,9) and (7,10)-entries are  $\frac{319}{5760(5-4q)}$ . Also, the (4,13) and (8,14)-cells are  $\frac{107}{3456(5-4q)}$  and the (12,15)-entry is  $\frac{197}{8640(5-4q)}$ .

The sixfold eigenvalue  $\frac{1}{20}$  now takes the more general form,  $\frac{1}{16} - \frac{1}{16(5-4q)}$ , the isolated eigenvalue  $\frac{1277}{21600}$  becomes  $\frac{1}{16} - \frac{73}{4320(5-4q)}$ , while the twofold eigenvalue  $\frac{539}{8640}$  is transformed to  $\frac{1}{16} - \frac{1}{1728(5-4q)}$ . Additionally, the threefold eigenvalue  $\frac{2327}{32400}$  becomes  $\frac{1}{16} + \frac{151}{3240(5-4q)}$ , the twofold eigenvalue  $\frac{1039}{14400}$  changes to  $\frac{1}{16} + \frac{139}{2880(5-4q)}$ , the isolated eigenvalue  $\frac{1583}{21600}$  is converted to  $\frac{1}{16} + \frac{233}{4320(5-4q)}$  and the isolated eigenvalue  $\frac{299}{3600}$  is transformed to  $\frac{1}{16} + \frac{37}{360(5-4q)}$ . The associated eigenspaces were as reported above.

### C. $N = 4, m = 3$

We have begun to investigate this particular scenario and have established, among other items, that the (52,52)-entry of the  $64 \times 64$  mean density matrix is

$$\frac{43581 + 10q(2160q - 6283)}{172800(5 - 4q)(3 - 2q)}, \quad (33)$$

the (52,61)-cell is

$$\frac{5026 - 2675}{172800(5 - 4q)(3 - 2q)}, \quad (34)$$

while the (53,53)-entry is

$$\frac{58387 + 6q(6480q - 15979)}{311040(5 - 4q)(3 - 2q)}. \quad (35)$$

### D. $N = 2$

We have also for the case,  $N = 2$ , been able to obtain a *two*-parameter family of  $2^m \times 2^m$  mean density matrices by replacing the uniform measure of Życzkowski *et al* by an (asymmetric) Dirichlet (or beta) distribution (30), with its two parameters  $q_1$  and  $q_2$  not necessarily being equal. Nevertheless, the decompositions of  $2^m$ -dimensional Hilbert space into the associated eigenspaces of the mean density matrices appear to be of the very same nature as reported in sec. II E. If we do equate the two parameters ( $q_1 = q_2 \equiv q$ ), then for the case,  $m = 2$ , we obtain the isolated eigenvalue  $\frac{1-q}{2(3-2q)}$  and the threefold eigenvalue  $\frac{5-3q}{6(3-2q)}$ . For  $m = 3$ , we have two quadruplets,  $\frac{2-q}{4(3-2q)}$  and  $\frac{1-q}{4(3-2q)}$ .

Analogously to what was done in [21] for the  $2^m \times 2^m$  density matrices, one might investigate the nature of the Bures distance between two mean density matrices (for  $N > 2$ ) corresponding to nonidentical sets of values of the  $N$  parameter(s) of the Dirichlet distribution (30).

### E. $N > 4$

It seems quite natural to conjecture, based on the results of this section, that, in general, factors of the form  $(N + 1 - Nq)$  will appear in parallel results for  $N > 4$ . We remark, in this regard, that for the case  $N = 2, m = 4$ , the denominators of the three distinct eigenvalues of the  $16 \times 16$  mean density matrix are all proportional to  $(5 - 2q)(3 - 2q)$  (cf. (33) - (35)).

## IX. CONCLUDING REMARKS

Of course, it would be of interest to obtain analogs of the results presented above for additional values of  $N$  and/or  $m$ , and to explore areas of further possible application (cf. [26,27]). The relation of the orthonormal bases of  $N^m$ -dimensional Hilbert space that we have reported, to other bases, such as the Bell ( $N = 2, m = 2$ ) and Greenberger-Horne-Zeilinger ( $N = 2, m > 2$ ) ones, used in the processing of quantum information [47–49], merits investigation. (These latter bases correspond to *maximally entangled* states.) In this regard, we make the simple observation that in the  $N = 2, m = 2$  case, the eigenvectors found by Krattenthaler and Slater [20] separate into a triplet ( $M_{2,0} = 3$ ) and singlet ( $M_{2,1} = 1$ ), as with the Bell basis. The triplet (and, of course, the singlet) is known to be irreducible [42, p. 103]. We have, however, not formally established the irreducibility of the eigenspaces for the  $N > 3$  cases reported above (secs. III, IV, V and VI).

It would be desirable to better understand the significance of the process (selection rule) by which we have annihilated those off-diagonal entries in the averaged density matrices (based on tensor products incorporating  $3 \times 3$  density matrices) which are equal to a rational number divided by  $\pi$  (leaving undisturbed those off-diagonal entries which are simply rational numbers) and thereby obtained eigenvalues, the multiplicities of which correspond to the (permutationally-symmetrized) multiplets. In this regard, it would be of interest to determine the relations of the

approach taken here to the alternative one of Katriel, Paldus and Pauncz [31] (from whose article we have excerpted certain passages at the end of sec. IIE).

We should also note for the benefit of those who might desire to further pursue the lines of investigation followed here that our results have, in large part, been based on MATHEMATICA [50] computations, in particular, multiple integrations over multivariate polynomials in trigonometric functions (sines and cosines of single angles). However, since MATHEMATICA would not directly integrate the cumbersome expressions (at least in an acceptable amount of time), we found it necessary at each integration stage to reduce the problem to its simplest possible (univariate) form, saving auxilarily the unused information for the next integration step. In fact, it appeared necessary at several points to pursue even more subtle strategies in order to reduce the computational (space and time) burden. In any case, we have attempted to present the most extensive analyses within our capabilities of achieving. We should also observe that we have not investigated, to any major extent, the possible role for *numerical*, as opposed to exact or symbolic, integration methods.

## ACKNOWLEDGMENTS

I would like to express appreciation to the Institute for Theoretical Physics for computational support in this research, C. Krattenthaler and T. Brun for certain technical advice, and D. Eardley for a discussion concerning [4].

- 
- [1] W. K. Wootters, Found. Phys. 20, 1365 (1990).
  - [2] K. R. W. Jones, Ann. Phys. (NY) 207, 140 (1991).
  - [3] W. Band and J. L. Park, Found. Phys. 7, 705 (1977).
  - [4] K. Życzkowski, P. Horodecki, A. Sanpera, and M. Lewenstein, Phys. Rev. A 58, 883 (1998).
  - [5] P. B. Slater, J. Math. Phys. 37, 2682 (1996).
  - [6] P. B. Slater, J. Phys. A 29, L271 (1996).
  - [7] P. B. Slater, J. Math. Phys. 38, 2274 (1997).
  - [8] P. B. Slater, *Comparative Noninformativities of Quantum Priors Based upon Monotone Metrics*, quant-ph/9703012 (to appear in Phys. Lett. A).
  - [9] P. B. Slater, *Thermodynamics of Spin-1 Systems*, quant-ph/9802019.
  - [10] M. J. W. Hall, Phys. Lett. A 242, 123 (1998).
  - [11] M. J. W. Hall, *Volume of Classical and Quantum Ensembles: Geometric Approach to Entropy and Information*, quant-ph/9806013.
  - [12] D. Petz and G. Toth, Lett. Math. Phys. 27, 205 (1993).
  - [13] D. Petz and C. Sudár, J. Math. Phys. 37, 2662 (1996).
  - [14] M. Hübner, Phys. Lett. A 163, 239 (1992); 179, 226 (1993).
  - [15] S. L. Braunstein and C. M. Caves, Phys. Rev. Lett. 72, 3439 (1994).
  - [16] S. L. Braunstein and G. J. Milburn, Phys. Rev. A 51, 1820 (1995).
  - [17] L. C. Biedenharn and J. D. Louck, *Angular Momentum in Quantum Physics* (Addison-Wesley, Reading, 1981).
  - [18] R. F. Werner, Phys. Rev. A 40, 4277 (1989).
  - [19] R. Jozsa, *Quantum Effects in Algorithms*, quant-ph/9805086.
  - [20] C. Krattenthaler and P. B. Slater, *Asymptotic Redundancies for Universal Quantum Coding*, quant-ph/9612043.
  - [21] P. B. Slater, Phys. Lett. A 244, 35 (1998).
  - [22] P. B. Slater, J. Phys. A 29, L601 (1996).
  - [23] R. Jozsa, M. Horodecki, P. Horodecki, and R. Horodecki, *Universal Quantum Information Compression*, quant-ph/9805017.
  - [24] B. S. Clarke and A. R. Barron, IEEE Trans. Inform. Th. 36, 453 (1990).
  - [25] B. S. Clarke and A. R. Barron, J. Statist. Plann. Infer. 41, 37 (1994).
  - [26] R. Pauncz, *Spin Eigenfunctions: Construction and Use*, (Plenum, New York, 1979).
  - [27] R. Pauncz, *The Symmetric Group in Quantum Chemistry*, (CRC Press, Boca Raton, 1995).
  - [28] P. B. Slater, *Quantum Statistical Thermodynamics of Two-Level Systems*, quant-ph/9706013.
  - [29] D. J. Rowe, J. Math. Phys. 36, 1520 (1995).
  - [30] S. Ališauskas, in *Symmetries in Science VI: From the Rotation Group to Quantum Algebras*, edited by B. Gruber (Plenum, New York, 1993), p. 19.
  - [31] J. Katriel, J. Paldus, and R. Pauncz, Intl. J. Quant. Chem. 28, 181 (1985) [errata, 29, 171].
  - [32] R. D. Kent and M. Schlesinger, Comp. Phys. Commun. 43, 413 (1987).

- [33] R. D. Kent, M. Schlessinger, and P. S. Ponnappalli, Phys. Rev. A 39, 19 (1989).
- [34] R. D. Kent and M. Schlessinger, Phys. Rev. A 48, 4156 (1993).
- [35] M. S. Byrd, *The Geometry of  $SU(3)$* , physics/9708015.
- [36] M. S. Byrd and E. C. G. Sudarshan,  *$SU(3)$  Revisited*, physics/9803029.
- [37] M. Byrd, *Differential Geometry on  $SU(3)$  with Applications to Three State Systems*, math-ph/9807032 (to appear in J. Math. Phys.).
- [38] F. J. Bloore, J. Phys. A 9, 2059 (1976).
- [39] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. 80, 5239 (1998).
- [40] P. Horodecki, Phys. Lett. A 232, 333 (1997).
- [41] N. Linden and S. Popescu, *Bound Entanglement and Teleportation*, quant-ph/9807069.
- [42] W. Greiner, *Quantum Mechanics: Symmetries*, (Springer, Berlin, 1994).
- [43] E. Leader and E. Predazzi, *An Introduction to Gauge Theories and Modern Particle Physics*, (Cambridge, New York, 1996).
- [44] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A 223, 1 (1996).
- [45] K. Życzkowski and M. Kuś, J. Phys. A 27, 4235 (1994).
- [46] J. O. Berger and J. M. Bernardo, Biometrika 79, 25 (1992).
- [47] A. Zeilinger, H. J. Bernstein, and M. A. Horne, J. Mod. Opt. 41, 2375 (1994).
- [48] A. Zeilinger, Phys. World 11(3), 35 (1998).
- [49] N. Gisin and H. Bechmann-Pasquinucci, *Bell Inequality, Bell States and Maximally Entangled States for  $n$  qubits*, quant-ph/9804045.
- [50] S. Wolfram, *The Mathematica Book*, (Wolfram Media, Champaign, 1996).

## List of Figures

1	Nonmonotone nature of the indicator function (8), corresponding to the probability distribution (3) with $u = -2$ . . . . .	5
---	---	---